MEASUREMENT OF ANGULAR ACCELERATION OF A RIGID BODY USING LINEAR ACCELEROMETERS

by

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INTRODUCTION

The feasibility of using linear accelerometers to determine the angular acceleration of a rigid body in planar motion has been demonstrated by Mertz (1). The kinematic principles involved does not prevent the extension of this method to a general three-dimensional motion. It can be shown that, in theory, only 5 accelerometers are required to obtain explicit expressions of the 3 components of angular acceleration of the body on which the transducers are mounted. A sixth accelerometer is needed to provide all three linear acceleration components for a complete definition of the kinematics of the rigid body. The difficulties encountered are also described and the accuracy of the measurements made is estimated.

THEORETICAL DEVELOPMENT

The acceleration of a point P on a rigid body is given by:

$$\mathbf{A}_P = \ddot{\mathbf{R}} + \mathbf{a} + 2 \times \boldsymbol{\omega} \times \mathbf{v} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_P) + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho}_P \tag{1}$$

As shown in Figure 1,

$\ddot{\mathbf{R}}$ - acceleration of the body-fixed frame(xyz) with respect to the inertial reference frame (XYZ)
$\mathbf{a}$ = acceleration of the point P relative to the body-fixed frame
$\boldsymbol{\omega}$ = angular velocity of the body
$\dot{\boldsymbol{\omega}}$ = angular acceleration of the body
$\mathbf{v}$ = velocity of the point P relative to the body-fixed frame
$\boldsymbol{\rho}_P$ = distance of the point P from the origin of the body-fixed frame

In view of the rigid body assumption,

$$\mathbf{a} = 0$$

$$\mathbf{v} = 0$$

Thus,

$$\mathbf{A}_P = \ddot{\mathbf{R}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_P) + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho}_P \tag{2}$$

With reference to the body-fixed frame,

$$\underset{\sim}{R} = R_x \underset{\sim}{i} + R_y \underset{\sim}{j} + R_z \underset{\sim}{k}$$

$$\underset{\sim}{\omega} = \omega_x \underset{\sim}{i} + \omega_y \underset{\sim}{j} + \omega_z \underset{\sim}{k}$$

and

$$\underset{\sim}{\rho}_P = \rho_{xP} \underset{\sim}{i} + \rho_{yP} \underset{\sim}{j} + \rho_{zP} \underset{\sim}{k}$$

where the unit vectors for this frame are $\underset{\sim}{i}$, $\underset{\sim}{j}$, and $\underset{\sim}{k}$. The components of $\underset{\sim}{A}_P$ along the body-fixed axes are:

$$A_{xP} = \ddot{R}_x + \omega_y(\omega_x \rho_{yP} - \omega_y \rho_{xP}) - \omega_z(\omega_z \rho_{xP} - \omega_x \rho_{zP}) + \dot{\omega}_y \rho_{zP} - \dot{\omega}_z \rho_{yP} \quad (3a)$$

$$A_{yP} = \ddot{R}_y + \omega_z(\omega_y \rho_{zP} - \omega_z \rho_{yP}) - \omega_x(\omega_x \rho_{yP} - \omega_y \rho_{xP}) + \dot{\omega}_z \rho_{xP} - \dot{\omega}_x \rho_{zP} \quad (3b)$$

$$A_{zP} = \ddot{R}_z + \omega_x(\omega_z \rho_{xP} - \omega_x \rho_{zP}) - \omega_y(\omega_y \rho_{zP} - \omega_z \rho_{yP}) + \dot{\omega}_x \rho_{yP} - \dot{\omega}_y \rho_{xP} \quad (3c)$$

The angular acceleration components about the body-fixed axes can be determined directly if the 5 accelerometers are located at three points in the configuration shown in Figure 1. At the origin of the body-fixed frame, 0, a pair of accelerometers are located such that their sensitive axes are in the direction of the y- and z- axes. At point 2, along the x-axis, another pair of accelerometers is placed in the y- and z- directions. A single accelerometer in the z-direction is located at point 1 along the y-axis. The sixth accelerometer should be directed along the x-axis and can be conveniently placed at the origin. Thus,

$$\ddot{R}_x = A_{x0} \qquad \text{and} \qquad \underset{\sim}{\rho}_1 = \rho_{y1} \underset{\sim}{j}$$

$$\ddot{R}_y = A_{y0} \qquad \underset{\sim}{\rho}_2 = \rho_{x2} \underset{\sim}{i}$$

$$\ddot{R}_z = A_{z0} \qquad \underset{\sim}{\rho}_0 = 0$$

From Equation 3c, for P = 1 and 2

$$A_{z1} - A_{z0} = \omega_y \omega_z \rho_{y1} + \dot{\omega}_x \rho_{y1} \tag{4}$$

$$A_{z2} - A_{z0} = \omega_x \omega_z \rho_{x2} - \dot{\omega}_y \rho_{x2} \tag{5}$$

and from Equation 3b, for P = 2

$$A_{y2} - A_{y0} = \omega_x \omega_y \rho_{x2} + \dot{\omega}_z \rho_{x2} \tag{6}$$

Upon rearranging Equations 4, 5, and 6,

$$\dot{\omega}_x = (A_{z1} - A_{z0})/\rho_{y1} - \omega_y \omega_z \tag{7}$$

$$\dot{\omega}_y = -(A_{z2} - A_{z0})/\rho_{x2} + \omega_x \omega_z \tag{8}$$

$$\dot{\omega}_z = (A_{y2} - A_{y0})/\rho_{x2} - \omega_x \omega_y \tag{9}$$

It can be seen that the angular acceleration components in Equations 7 through 9 require prior knowledge of the angular velocities about these axes. It is thus necessary to perform a numerical integration of these equations to solve for the accelerations, assuming that the linear accelerations are known (measured) and the distances ρ_{y1} and ρ_{x2} are known to a high degree of accuracy. This stepwise integration procedure usually results in an accumulation of error in the values of $\underset{\sim}{\omega}$ which in turn affects the accuracy of $\underset{\sim}{\dot{\omega}}$. Thus, although the method is feasible in principle, it is difficult to determine the desired angular acceleration and velocity accurately as a result of dependence on values calculated at a previous time step. To obtain reliable angular acceleration results, the accuracy of the measured data is estimated to be approximately 0.1% of the peak linear acceleration. This degree of accuracy is not attainable with existing accelerometers which must withstand high linear accelerations.

AN ALTERNATE METHOD

An alternate method which can circumvent the difficulties encountered during numerical integration is to use nine linear accelerometers in the configuration shown in Figure 2. It consists of one set of triaxial accelerometers and three pairs of biaxial ones. With this arrangement, it is possible to obtain three more equations similar to Equations 7 through 9. They are:

$$\dot{\omega}_x = -(A_{y3} - A_{y0})/\rho_{z3} + \omega_y\omega_z \tag{10}$$

$$\dot{\omega}_y = (A_{x3} - A_{x0})/\rho_{z3} - \omega_z\omega_x \tag{11}$$

$$\dot{\omega}_z = -(A_{x1} - A_{x0})/\rho_{y1} + \omega_x\omega_y \tag{12}$$

By eliminating the cross products of the angular velocity components from Equations 7 through 12, the angular acceleration can be determined directly at each time point, without reliance on values of parameters computed at the previous time step. The equations for angular acceleration take the form:

$$\dot{\omega}_x = (A_{z1} - A_{z0})/2\rho_{y1} - (A_{y3} - A_{y0})/2\rho_{z3} \tag{13}$$

$$\dot{\omega}_y = (A_{x3} - A_{x0})/2\rho_{z3} - (A_{z2} - A_{z0})/2\rho_{x2} \tag{14}$$

$$\dot{\omega}_z = (A_{y2} - A_{y0})/2\rho_{x2} - (A_{x1} - A_{x0})/2\rho_{y1} \tag{15}$$

Similarly, by eliminating the angular acceleration terms from Equations 7 through 12, the angular velocity components can be determined from a set of non-linear algebraic equations:

$$\omega_y\omega_z = (A_{z1} - A_{z0})/2\rho_{y1} + (A_{y3} - A_{y0})/2\rho_{z3} \tag{16}$$

$$\omega_z\omega_x = (A_{x3} - A_{x0})/2\rho_{z3} + (A_{z2} - A_{z0})/2\rho_{x2} \tag{17}$$

$$\omega_x\omega_y = (A_{y2} - A_{y0})/2\rho_{x2} + (A_{x1} - A_{x0})/2\rho_{y1} \tag{18}$$

VALIDATION OF THE 9-ACCELEROMETER METHOD

To demonstrate that this method of computing angular acceleration and velocity is practical and reliable, it is necessary to perform a two-step validation procedure. Firstly, the method must yield consistent and accurate results when fictitious data representing a known motion are used. Secondly, experimental data acquired from a known rotation of a rigid body should be used to validate the method.

A set of fictitious data representing rotation about the y-axis was created to demonstrate the stability of the method. For simplicity, the 'experimentally' measured accelerations were assumed to be sinusoidal functions of time. A_{x3}, A_{z0}, and A_{z1} took the form of $A_e(t)$, as shown in Figure 3. For a pure rotation about the y-axis, it is necessary that A_{y0}, A_{y2}, and A_{y3} remain quiescent and convenient to set

$$A_{x0} = A_{z2} = A_{x1} = A_e(t)/2 \tag{19}$$

With these input values, it was found that $\dot{\omega}_x = \dot{\omega}_z = 0$ and $\omega_x = \omega_z = 0$ throughout the duration of the run and that the only non-zero quantities were $\dot{\omega}_y$ and ω_y.

In order to demonstrate the stability of this method of solution, errors were deliberately introduced into the input data. For example, if

$$A_{y2} = -A_{y3} = 0.05A_e(t) \tag{20}$$

as shown by the dotted line in Figure 3, the angular acceleration components about the y-axis were at least two orders of magnitude larger than those about the other two axes, using the 9-accelerometer approach. This is shown in Figure 4. If only six accelerometers were used, the values about the x and z- axes very quickly reached the same order of magnitude as those about the y-axis.

If the error was reduced to 0.1%, the 6-accelerometer solution was marginally stable but the output was worse than the 9-accelerometer solution with a $\pm$ 5% error.

Validation of this method with experimental data necessitated the acquisition of such data, using an experimental set-up shown in Figure 5. It consists of a rigid plate mounted on the shaft of a motor. An accelerometer mount was designed to be attached to this plate in a variety of orientations so that the accelerometers could experience a general three-dimensional motion when the rigid body executed a planar rotation. The plate was given an initial angular displacement and released. It could either impact a stop or oscillate freely to yield the desired data. A validation run was made with the accelerometer mount axes inclined at 45° to the shaft of the rigid body (y-axis). The result is encouraging. A plot of the angular acceleration components is shown in Figure 6. Despite the presence of cross-axis sensitivity which can be as high as 5%, The magnitudes of the three components are approximately the same.

Note that the solution of the non-linear algebraic equations 16 through 18 is difficult if the measured values on the righthand side contain errors. If inconsistencies in sign or magnitude are encountered, the angular velocity is determined by means of an integration procedure, for that time point.

DISCUSSION AND CONCLUSIONS

This paper demonstrates the practical difficulty encountered in the computation of angular acceleration and velocity using data from six linear accelerometers. A nine-accelerometer scheme has been proposed to circumvent the problem in a simple and straight-forward manner. The choice of the location of these accelerometers is the key to the solution of this problem. The configuration shown in Figure 2 is probably the optimal arrangement for the algebraic determination of angular acceleration and velocity. The method is stable and tolerates errors as large as 5%. The six-accelerometer approach can only tolerate errors of 0.1%.

An experimental set-up was used to validate this method. Preliminary results show that the proposed nine-accelerometer method is feasible. Based on the experience gained from both the experimental data and those simulating rigid body motion, it is concluded that if a set of nine linear acceleration measurements are made accurately, the angular acceleration, velocity and displacement of a rigid body can be computed.

ACKNOWLEDGMENTS

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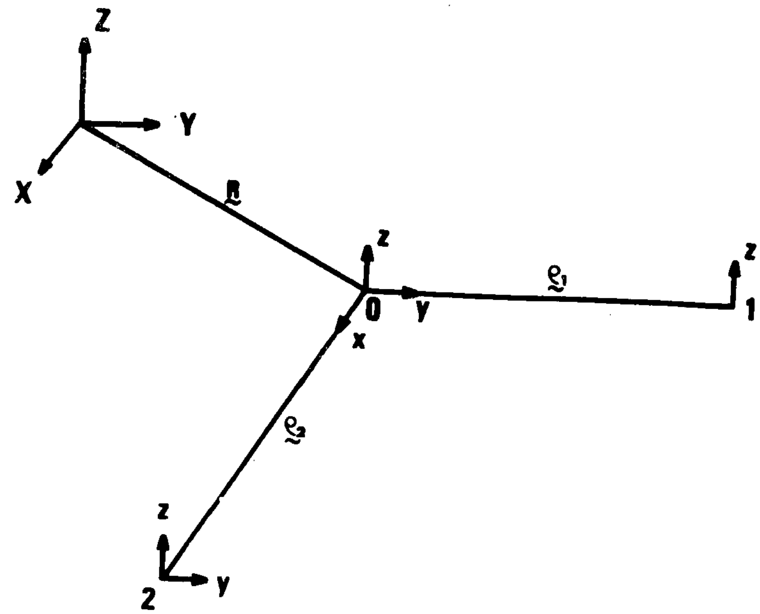

FIGURE 1. ACCELEROMETER LOCATIONS FOR THE SIX-ACCELEROMETER SCHEME.

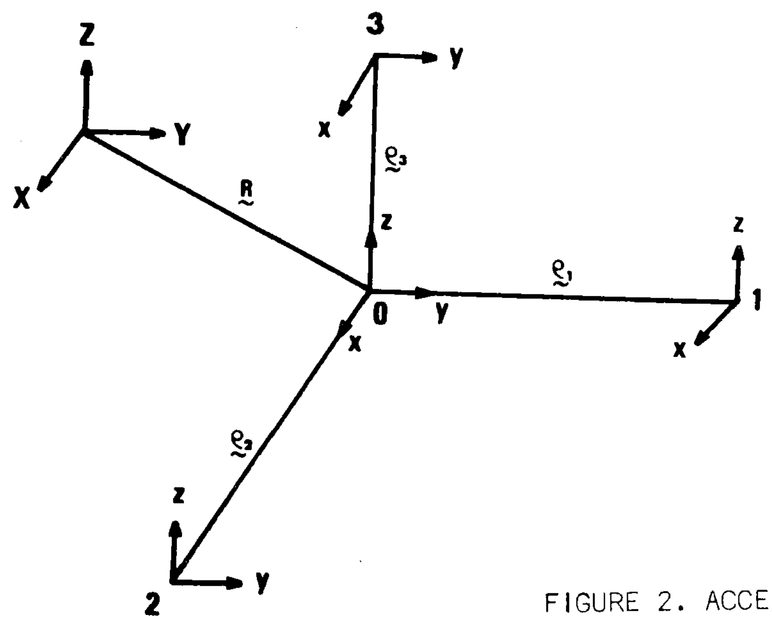

FIGURE 2. ACCELEROMETER LOCATIONS FOR THE NINE-ACCELEROMETER SCHEME.

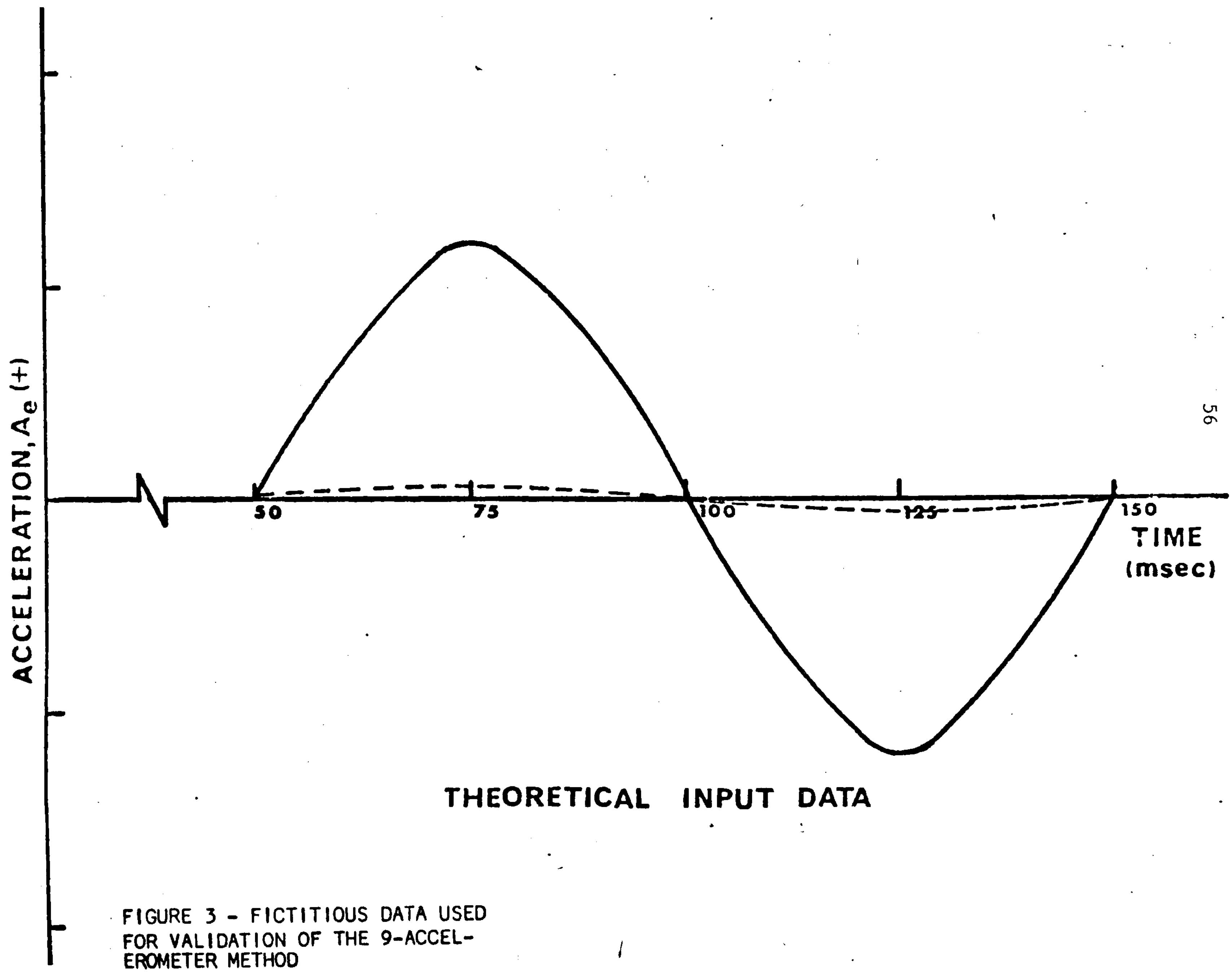

FIGURE 3 - FICTITIOUS DATA USED FOR VALIDATION OF THE 9-ACCELEROMETER METHOD

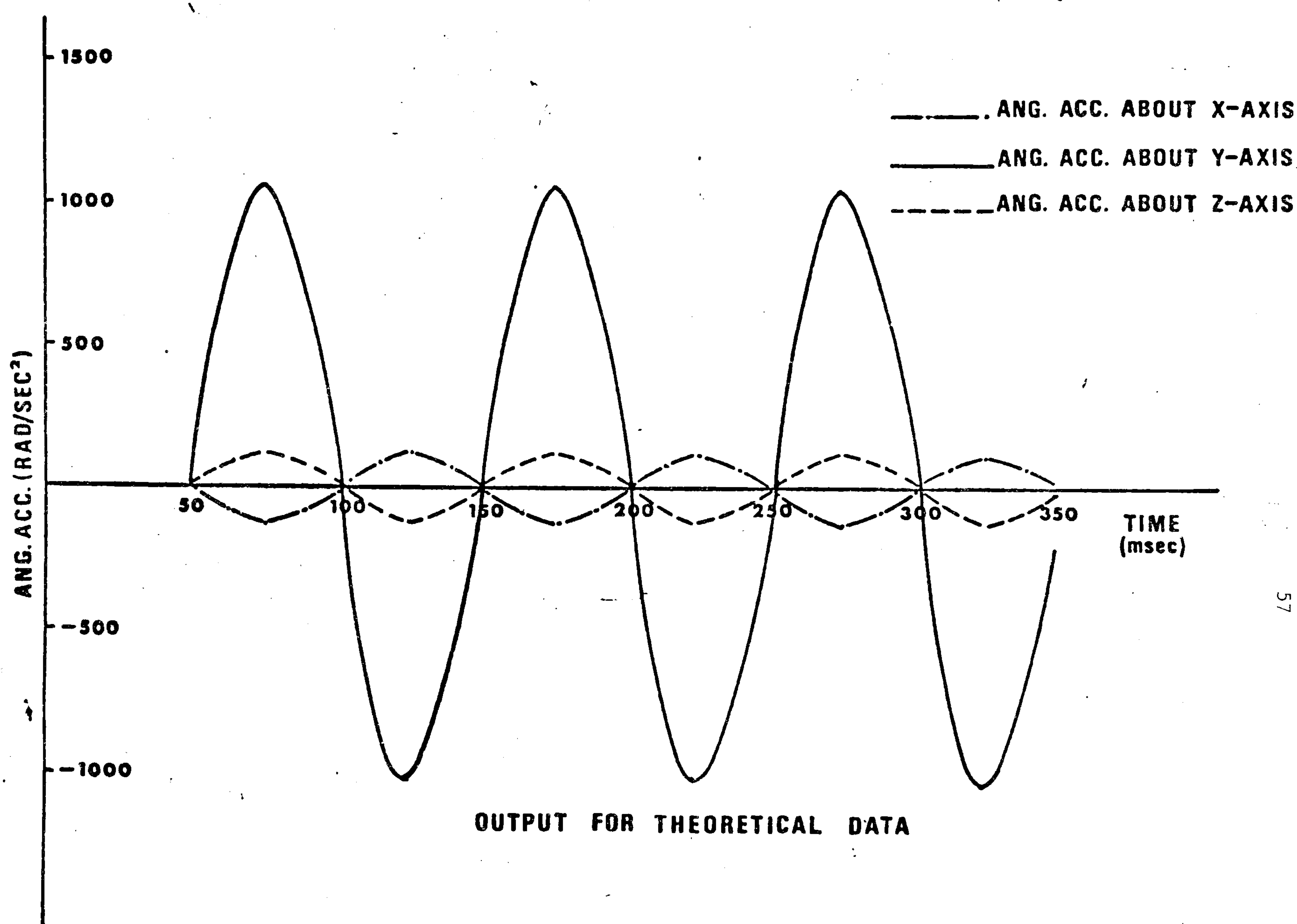

FIGURE 4 - COMPUTED ANGULAR ACCELERATIONS USING FICTITIOUS DATA CONTAINING 5% ERRORS

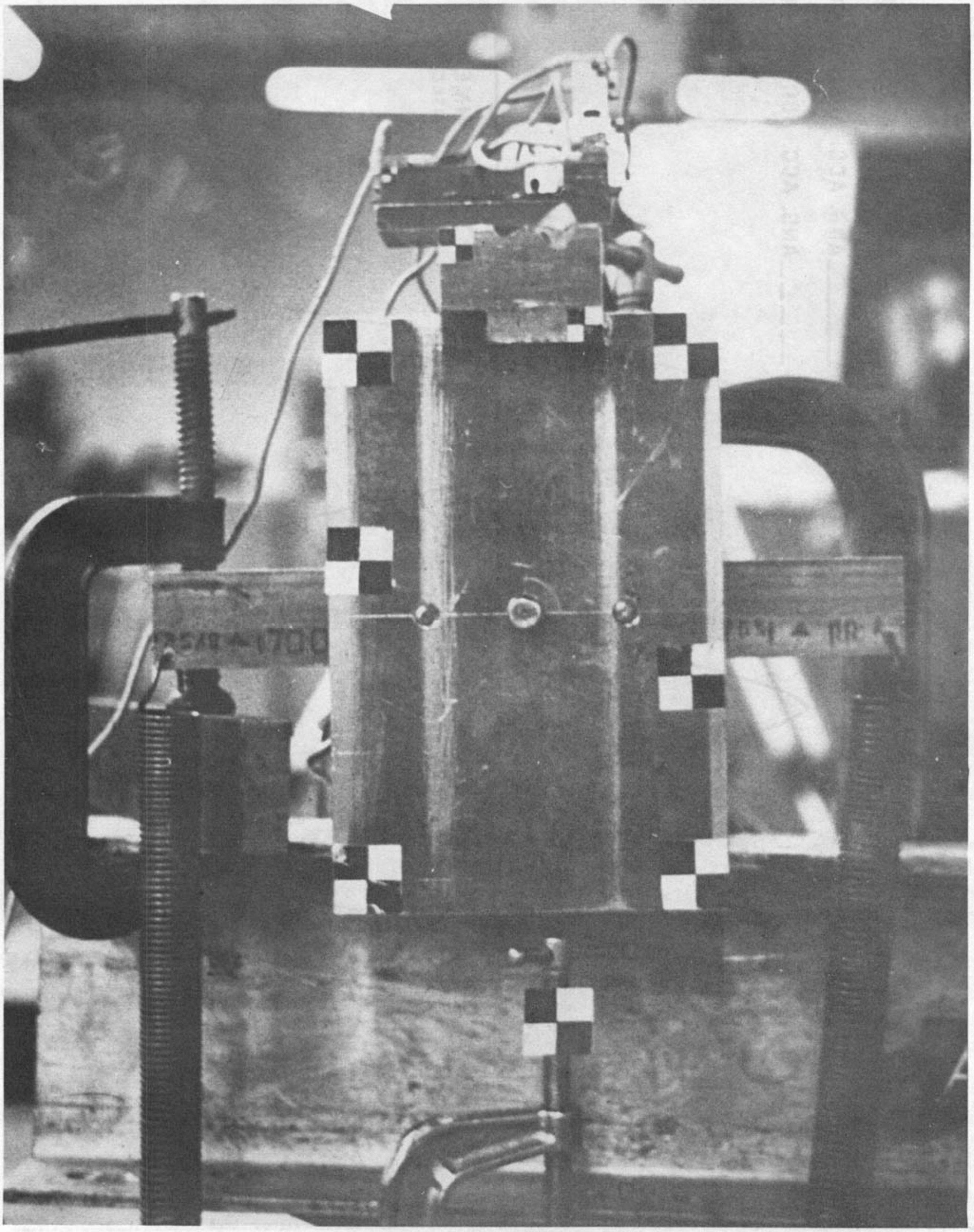

Fig. 5: Experimental set-up used to validate the 9-accelerometer method.

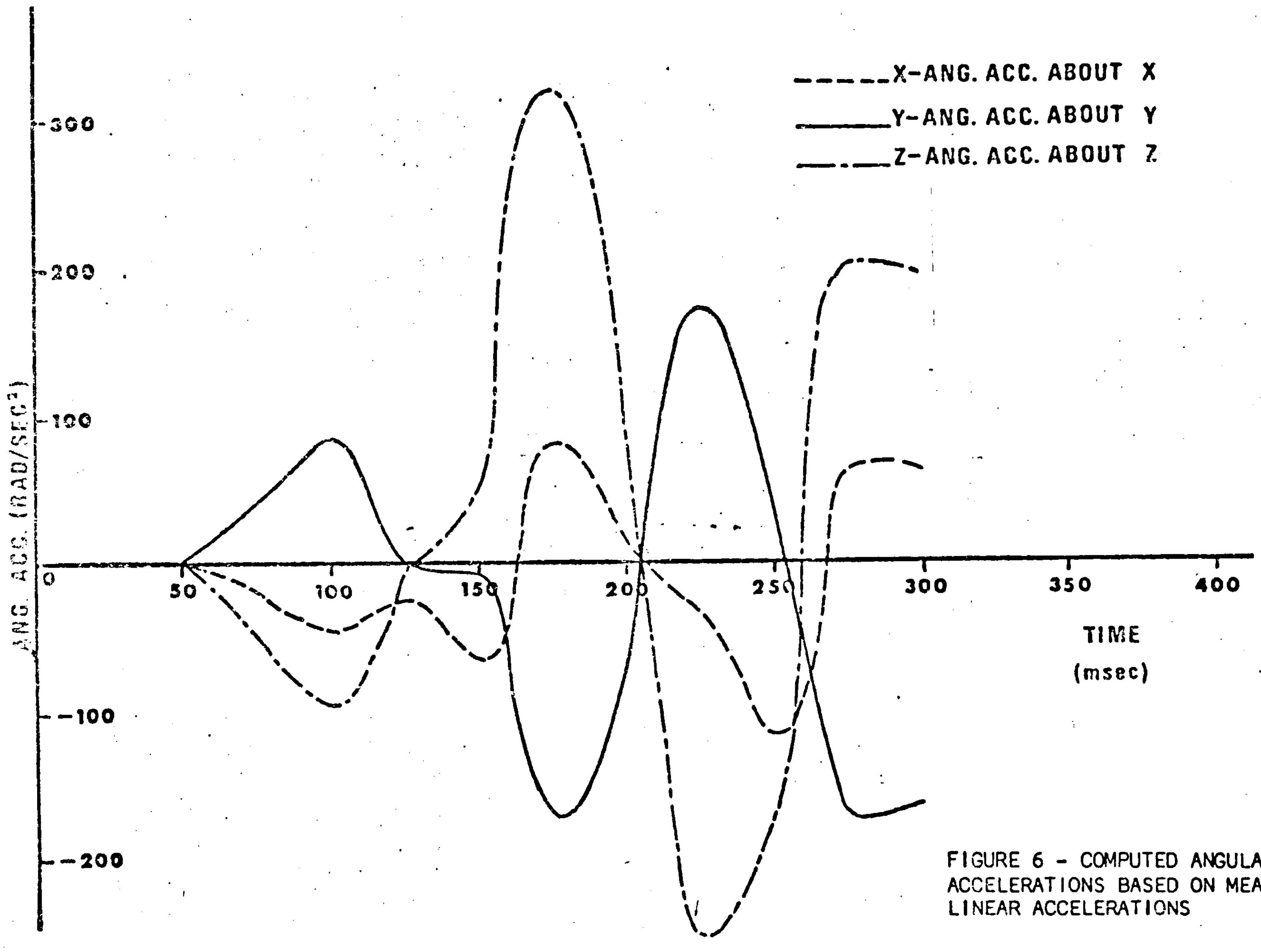

FIGURE 6 - COMPUTED ANGULAR ACCELERATIONS BASED ON MEASURED LINEAR ACCELERATIONS

OUTPUT FOR 45° OBLIQUE MODE